Derivation of Linearized Error State Dynamics

Let the state of the vehicle at time t be

$$\boldsymbol{x}_t = \begin{bmatrix} x_t \\ y_t \\ \vartheta_t \\ v_t \end{bmatrix}, \tag{1}$$

which evolves according to the discrete-time dynamics

$$\boldsymbol{x}_{t+1} = f(\boldsymbol{x}_t, \boldsymbol{u}_t) = \begin{bmatrix} x_t + v_t \cos\vartheta_t \Delta t \\ y_t + v_t \sin\vartheta_t \Delta t \\ \vartheta_t + \omega_t \Delta t \\ v_t + a_t \Delta t \end{bmatrix}, \tag{2}$$

where

$$\boldsymbol{u}_t = \begin{bmatrix} \omega_t \\ a_t \end{bmatrix} \tag{3}$$

is the control input. Given some reference control signal $\boldsymbol{u}_t^*$ and reference trajectory

$$\boldsymbol{x}_{t+1}^* = f\left(\boldsymbol{x}_t^*, \boldsymbol{u}_t^*\right), \tag{4}$$

define the error state

$$\Delta\boldsymbol{x}_t = \boldsymbol{x}_t - \boldsymbol{x}_t^* \tag{5}$$

and error controls

$$\Delta\boldsymbol{u}_t = \boldsymbol{u}_t - \boldsymbol{u}_t^*. \tag{6}$$

The dynamics of the error state are

$$\begin{aligned} \Delta\boldsymbol{x}_{t+1} = g(\Delta\boldsymbol{x}_t, \Delta\boldsymbol{u}_t) &= \begin{bmatrix} x_t + v_t \cos\vartheta_t \Delta t - x_t^* - v_t^* \cos\vartheta_t^* \Delta t \\ y_t + v_t \sin\vartheta_t \Delta t - y_t^* - v_t^* \sin\vartheta_t^* \Delta t \\ \vartheta_t + \omega_t \Delta t - \vartheta_t^* - \omega_t^* \Delta t \\ v_t + a_t \Delta t - v_t^* - a_t^* \Delta t \end{bmatrix} & (7) \\ &= \begin{bmatrix} \Delta x_t + \left(\left(v_t^* + \Delta v_t\right) \cos\left(\vartheta_t^* + \Delta\vartheta_t\right) - v_t^* \cos\vartheta_t^*\right) \Delta t \\ \Delta y_t + \left(\left(v_t^* + \Delta v_t\right) \sin\left(\vartheta_t^* + \Delta\vartheta_t\right) - v_t^* \sin\vartheta_t^*\right) \Delta t \\ \Delta\vartheta_t + \Delta\omega_t \Delta t \\ \Delta v_t + \Delta a_t \Delta t \end{bmatrix} & (8) \\ &= \begin{bmatrix} \Delta x_t + \left(v_t^* \cos\left(\vartheta_t^* + \Delta\vartheta_t\right) + \Delta v_t \cos\left(\vartheta_t^* + \Delta\vartheta_t\right) - v_t^* \cos\vartheta_t^*\right) \Delta t \\ \Delta y_t + \left(v_t^* \sin\left(\vartheta_t^* + \Delta\vartheta_t\right) + \Delta v_t \sin\left(\vartheta_t^* + \Delta\vartheta_t\right) - v_t^* \sin\vartheta_t^*\right) \Delta t \\ \Delta\vartheta_t + \Delta\omega_t \Delta t \\ \Delta v_t + \Delta a_t \Delta t \end{bmatrix}. & (9) \end{aligned}$$

Taking the Taylor series expansion of $g(\Delta\boldsymbol{x}_t, \Delta\boldsymbol{u}_t)$ about $\Delta\boldsymbol{x}_t = 0$ and $\Delta\boldsymbol{u}_t = 0$ (i.e., about the reference trajectory) and truncating after the first-order terms yields the linearized dynamic system

$$\Delta\boldsymbol{x}_{t+1} = A_t \Delta\boldsymbol{x}_t + B_t \Delta\boldsymbol{u}_t, \tag{10}$$

where

$$A_t = \frac{\partial g}{\partial \Delta\boldsymbol{x}_t}(0,0) = \begin{bmatrix} 1 & 0 & \frac{\partial x_{t+1}}{\partial \Delta\vartheta_t}(0,0) & \frac{\partial x_{t+1}}{\partial \Delta v_t}(0,0) \\ 0 & 1 & \frac{\partial y_{t+1}}{\partial \Delta\vartheta_t}(0,0) & \frac{\partial y_{t+1}}{\partial \Delta v_t}(0,0) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{11}$$

$$\frac{\partial x_{t+1}}{\partial \Delta\vartheta_t} = \left(-v_t^* \sin\left(\vartheta_t^* + \Delta\vartheta_t\right) - \Delta v_t \sin\left(\vartheta_t^* + \Delta\vartheta_t\right)\right) \Delta t \implies \frac{\partial x_{t+1}}{\partial \Delta\vartheta_t}(0,0) = -v_t^* \sin\vartheta_t^* \Delta t\,, \tag{12}$$

$$\frac{\partial x_{t+1}}{\partial \Delta v_t} = \cos\left(\vartheta_t^* + \Delta\vartheta_t\right) \Delta t \implies \frac{\partial x_{t+1}}{\partial \Delta v_t}(0,0) = \cos\vartheta_t^* \Delta t\,, \tag{13}$$

$$\frac{\partial y_{t+1}}{\partial \Delta\vartheta_t} = \left(v_t^* \cos\left(\vartheta_t^* + \Delta\vartheta_t\right) + \Delta v_t \cos\left(\vartheta_t^* + \Delta\vartheta_t\right)\right) \Delta t \implies \frac{\partial y_{t+1}}{\partial \Delta\vartheta_t}(0,0) = v_t^* \cos\vartheta_t^* \Delta t\,, \tag{14}$$

$$\frac{\partial y_{t+1}}{\partial \Delta v_t} = \sin\left(\vartheta_t^* + \Delta\vartheta_t^*\right) \Delta t \implies \frac{\partial y_{t+1}}{\partial \Delta v_t}(0,0) = \sin\vartheta_t^* \Delta t\,, \tag{15}$$

$$\tag{16}$$

$$\therefore A_t = \begin{bmatrix} 1 & 0 & -v_t^* \sin\vartheta_t^* \Delta t & \cos\vartheta_t^* \Delta t \\ 0 & 1 & v_t^* \cos\vartheta_t^* \Delta t & \sin\vartheta_t^* \Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{17}$$

and

$$B_t = \frac{\partial g}{\partial \Delta \boldsymbol{u}_t}(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \Delta t & 0 \\ 0 & \Delta t \end{bmatrix}. \tag{18}$$

If the error control signal $\Delta\boldsymbol{u}_t$ is computed based on the linearized system in eq. (10), the true state trajectory will be

$$\boldsymbol{x}_{t+1} = f\left(\boldsymbol{x}_t, \Delta\boldsymbol{u}_t + \boldsymbol{u}_t^*\right)\ . \tag{19}$$